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REFERENCES

[1] G. F. Franklin, J. D. Powell, and M. L. Workman, *Digital Control of Dynamic Systems*, 2nd ed. Reading, MA: Addison-Wesley, 1990.
[2] M. Gevers and G. Li, *Parameterizations in Control, Estimation and Filtering Problems*. Berlin, Germany: Springer–Verlag, 1993.
[3] H. Hanselmann, "Implementation of digital controllers. A survey," *Automatica*, vol. 23, pp. 7–32, 1987.
[4] S. Y. Hwang, "Minimum uncorrelated unit noise in state space digital filtering," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 25, pp. 273–281, Aug. 1977.
[5] M. A. Rotea and D. Williamson, "Optimal realizations of finite wordlength digital filters and controllers," *IEEE Trans. Circuits Syst.*, vol. 42, pp. 61–72, Feb. 1995.
[6] D. Williamson, *Digital Control and Implementation. Finite Wordlength Considerations*. Australia: Prentice–Hall, 1991.
[7] G. Zhu, K. M. Grigoriadis, and R. E. Skelton, "Covariance control design for the Hubble space telescope," *AIAA J. Guid., Control Dynam.*, vol. 18, pp. 230–236, 1995.

The L^2-Optimal Time-Delay Rational Laplace Model Revisited

P. Vilbé, S. Azou, P. Bréhonnet, and L. C. Calvez

***Abstract*—An optimization technique is provided for the approximation of the pure time delay. Owing to a near-optimal starting point, the integral square error is reduced through few iterations of a Gauss–Newton process. Compact implementation and robustness should encourage a large use in various engineering domains.**

***Index Terms*— Gauss–Newton optimization, mathematical techniques, pure time-delay model.**

I. INTRODUCTION

Electronic circuits frequently involve pure time-delay components, or include an active subsystem reacting as a pure delay or are modeled as pure delay connected with a linear or nonlinear model. Mathematical time-delay models used today in various software tools are based on the Padé approximation [1]. This model is easy to implement but, obtained from a Mac-Laurin series expansion of e^{-s} about zero, it is known that the approximation is quite accurate only for great values of t. For transient response analysis the small time behavior must not be neglected. Various suboptimal approximations are available, but this work focuses on the L^2-optimal approximation of the time delay over $[0, \infty[$.

In engineering applications, optimization procedures are sometimes dreadful but, when carefully mastered, they become a powerful tool (root finding is a well-known example). Making a secure starting procedure based on a suitable initial parameter set is a nonobvious problem. For engineering purposes, an autoboot procedure without any human intervention for any usual required order is looked for. It will be shown here that a clever implementation easily yields the L^2-optimal model.

The first section summarizes previous results about the approximation of a known signal with irrational Laplace transform. Efficiency of the optimization process for the time delay modeling relies on some remarks introduced in Section III. The last section analyzes the quality of the proposed procedure.

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II. BACKGROUND

Modeling the unit time delay is formulated as the minimization of the scalar cost function $Q \triangleq \langle e, e\rangle$ where $e = \tilde{f} - f$ reflects the discrepancy between the response $\tilde{f}(t)$ of the unknown model and the ideal rectangular pulse $f(t) = U(t) - U(t-1)$

$$Q \triangleq \langle e, e\rangle, \text{ with } e(t) = \sum_{i=1}^{n} r_i e^{-b_i t} - f(t)$$

and

$$\begin{cases} \mathcal{R}e(b_i) > 0, & i = 1 \cdots n \\ b_i \neq b_j, & \text{for } i \neq j \end{cases} \tag{1}$$

where $\langle f, g\rangle$ stands for the scalar product of two complex valued functions and is defined as $\langle f, g\rangle \triangleq \int_0^\infty f(t)g^*(t)\,dt$.

As seen in [2], approximation with exponential sums has been extensively investigated in the past. However, focusing on the specific problem of time delay modeling, present work takes advantage of more recent development [3], [4].

Dealing with the representation of signals with known Laplace transforms $\hat{f}(s)$, pioneering works [5], [6] have shown that this approximation problem in the time domain is transformed into an interpolation problem in the Laplace domain and the L^2-optimal solution satisfies the following basic equations:

$$\begin{cases} \hat{e}(s)\,|_{s=b_i} = 0, \\ \hat{e}'(s)|_{s=b_i} = 0, \end{cases} \qquad \text{with } \hat{e}(s) = \hat{\tilde{f}}(s) - \hat{f}(s).$$

It is known that the L^2-optimal residues derive from the standard least squares theory and thus are obtained as the solution of a set of linear equations. It will be seen further, that for the present problem constraining the parameters is required. Thus, with Calvez *et al.* [3] the optimal vector of residues r_i is written as

$$\mathbf{r} = \mathbf{M}(\mathbf{f}_{\boldsymbol{\psi}} + \boldsymbol{\chi}) \text{ with } \begin{cases} \mathbf{M} = \left(\boldsymbol{\Psi}^{\mathbf{T}}\right)^{-\mathbf{1}} = \mathbf{C}^{\mathbf{T}}\mathbf{C}^{*} \\ \boldsymbol{\chi} = \mathbf{L}^{\mathbf{H}}\left(\mathbf{L}\mathbf{M}\mathbf{L}^{\mathbf{H}}\right)^{-\mathbf{1}}(\mathbf{y} - \mathbf{L}\mathbf{M}\mathbf{f}_{\boldsymbol{\psi}}) \end{cases} \tag{2}$$

where the superscript $*$ stands for the conjugate and H for the conjugate transpose.

Comments.

1) From the classical L^2 theory it is known that the unconstrained solution is given by $\boldsymbol{\Psi}^{\mathbf{T}}\mathbf{r} = \mathbf{f}_{\boldsymbol{\psi}}$ where $\boldsymbol{\Psi}$ denotes the $n \times n$ Gram matrix with element (i, j) equal to $\langle \psi_i, \psi_j\rangle$. At this stage, the set of approximating functions is composed of the n distinct exponentials $\psi_i(t) = e^{-b_i t}$ and vector $\mathbf{f}_{\boldsymbol{\psi}}$ denotes the vector with ith component $f_{\psi_i} = \langle f, \psi_i\rangle$.
2) When an orthogonalization process of the basis functions is available, the second expression $\mathbf{C}^{\mathbf{T}}\mathbf{C}^{*}$ for $\mathbf{M}$ is used to get a closed-form formula, avoiding possible numerical inversion problems.

3) The standard unconstrained L^2 solution $\mathbf{M}\mathbf{f}_{\boldsymbol{\psi}}$ is corrected with the additive contribution $\mathbf{M}_{\boldsymbol{\chi}}$ introduced when the linear constraint equation $\mathbf{L}\mathbf{r} = \mathbf{y}$ must be satisfied. In most engineering problems the number of constraints is a small integer, thus the rank l of the square matrix $\mathbf{L}\mathbf{M}\mathbf{L}^{\mathbf{H}}$ is small and the computation of the required inverse is a very simple task. In the sequel, at least one constraint equation will always be set in order to preserve the low-pass filter characteristic of the model.

In contrast, the problem is nonlinear in the b_i and improving the exponential exponents requires linearization. Let $\boldsymbol{\delta} = (\Delta r_1, \Delta r_2, \cdots, \Delta r_n, \Delta b_1, \Delta b_2, \cdots, \Delta b_n)$ denote a vector of small corrections which, neglecting second-orders terms, yields the first-order changes $\Delta\tilde{f} = \sum_{i=1}^{2n} \delta_i \overline{\psi}_i$ with

$$\overline{\psi}_i(t) \triangleq \begin{cases} \psi_i(t), & i = 1 \cdots n \\ \psi'_{i-n}(t) = \dfrac{\partial \tilde{f}(t)}{\partial b_i} = -r_i t e^{-b_i t}, & i = (n+1) \cdots 2n. \end{cases}$$

After correction, the new error is $e_\Delta = \tilde{f} + \Delta\tilde{f} - f = \Delta\tilde{f} - (-e)$ and the δ_i which minimizes the energy $Q_\Delta = \langle e_\Delta, e_\Delta \rangle$ can be seen as the coefficients of the best representation of $-e$ in terms of the extended basis functions $\{\overline{\psi}_i\} \triangleq \{e^{-b_1 t}, \cdots, e^{-\gamma_b t}, -r_1 t e^{-b_1 t}, \cdots, -r_n t e^{-b_n t}\}$. Similarly to (2) the best coefficients δ_i are obtained from the standard least-squares theory as

$$\boldsymbol{\delta} = \overline{\mathbf{M}}\left(-\mathbf{e}_{\overline{\boldsymbol{\psi}}} + \overline{\boldsymbol{\chi}}\right) \text{ with } \cdot \begin{cases} \overline{\mathbf{M}} = \left(\overline{\boldsymbol{\Psi}}^{H}\right)^{-1} = \overline{\mathbf{C}}^{\mathbf{T}}\overline{\mathbf{C}}^{*} \\ \overline{\boldsymbol{\chi}} = \overline{\mathbf{L}}^{\mathbf{H}}\left(\overline{\mathbf{L}}\,\overline{\mathbf{M}}\,\overline{\mathbf{L}}^{\mathbf{H}}\right)^{-1}\left(\overline{\mathbf{y}} + \overline{\mathbf{L}}\, s\overline{\mathbf{M}} e_{\overline{\boldsymbol{\psi}}}\right), & \overline{\mathbf{y}} = 0 \end{cases} \tag{3}$$

where the n lower components of $\boldsymbol{\delta}$ (denoted by the vector $\boldsymbol{\delta}_l$) are used as the progressing step for the b_i parameters. Thus, for the $(k+1)$th iteration $\mathbf{b}^{(k+1)}$ is written as

$$\mathbf{b}^{(k+1)} = \mathbf{b}^{(k)} + \boldsymbol{\delta}_l.$$

Comments.

1) Since [7] it is known that a such correction gives rise to a Gauss–Newton optimization process.
2) Managing linear constraint equations on the parameters r_i, b_i requires the following extension:

$$\overline{\mathbf{L}}\boldsymbol{\delta} = \mathbf{0}$$

with

$$\overline{l}_{ij} \triangleq \begin{cases} l_{ij}, & j = 1 \cdots n \\ l'_{ij} = r_{j-n} \dfrac{\partial l_{ij-n}(\mathbf{b})}{\partial b_{j-n}}, & j = (n+1) \cdots 2n \end{cases} \tag{4}$$

which must not be omitted even when the r_i are the only parameters subject to constraints.
3) As above, owing to a possible orthogonalization process of the $2n$ functions of the extended basis, a closed-form inverse can be used.

III. Numerical Computations

Beyond the shortness and convenience of the preceding expressions, the optimization process requires a careful and clever implementation, noticing the mathematical validity hypotheses and finite numerical accuracy. In short, the most important hypotheses can be summarized as follows.

1) The corrective term $\boldsymbol{\delta}$ is significant only in the neighborhood of a local extrema.
2) The concept of a set of approximating functions requires that the functions be linearly independent.
3) The quadratic criterion holds only if the basis functions are energy bounded.

Remark 1: The first item sets the hard problem of finding a starting point for the iterative process. The procedure must boot with an arbitrary solution, sufficiently close to the unknown optimal solution. If a high-quality starting solution is not systematically found, then a special and robust starting iterative procedure (gradient for instance) must be executed first, before commuting and locking to the efficient Gauss–Newton iterations. However, such a process is rather cumbersome and tedious and it will be avoided in the present particular case.

As specified in the second item, assuming linear independence when modeling various signals gives rise to an important difficulty. It must be considered that, during the migration of the poles through the complex plane, their type may change, i.e., a pair of complex poles may become two real poles or vice versa. Thus, there is a possible crossing by a double pole, which generates a new function to be added to the previous set. Such work has been done soon by Harman *et al.* [8]. On the opposite, Vilbé *et al.* [4] introduced approximating functions avoiding this heavy managing overhead. In the present case, it will be found that the proposed high quality starting set of poles avoids the merging poles process.

The third item means that, for each new set of poles the stability of all approximation functions has to be tested.

We now emphasize some numerical aspects, leading to the L^2-optimal models of delay described in the next section and corresponding to the Matlab scripts of the Appendix.

Computation of $\mathbf{M}$ *and* $\overline{\mathbf{M}}$*:* In most least squares problems dealing with high-order approximation, the closed forms for $\mathbf{M}$ and $\overline{\mathbf{M}}$ are welcome. However, since the $\boldsymbol{\Psi}$ and $\overline{\boldsymbol{\Psi}}$ matrices are Hermitian, a standard inversion algorithm taking advantage of this property yields reliable results as long as a quasi-double pole is not present. It should be noted that a merging pole would also give rise to numerical difficulties in an orthogonalization process, if a specific detection and treatment is not provided. Here the two following matrices will be inverted using the standard Matlab `inv` function:

$$\overline{\boldsymbol{\Psi}} = \begin{pmatrix} \boldsymbol{\Psi}_{\mathbf{11}} \boldsymbol{\Psi}_{\mathbf{12}} \\ \boldsymbol{\Psi}_{\mathbf{12}}^{\mathbf{H}} \boldsymbol{\Psi}_{\mathbf{22}} \end{pmatrix}, \qquad \boldsymbol{\Psi}_{\mathbf{11}} = \boldsymbol{\Psi} \tag{5}$$

with

$$\Psi_{11}(i,j) = \frac{1}{b_i + b_j^*}, \qquad \Psi_{12}(i,j) = \frac{-r_j^*}{\left(b_i + b_j^*\right)^2}$$
$$\Psi_{22}(i,j) = \frac{2 r_i r_j^*}{\left(b_i + b_j^*\right)^3}.$$

Projections Vectors $\mathbf{f}_{\boldsymbol{\psi}}$ *and* $\mathbf{e}_{\overline{\boldsymbol{\psi}}}$*:* Dealing with known Laplace transform signals, the projected vectors are readily obtained from the Laplace transform

$$f_{\psi_i} \triangleq \langle f, \psi_i \rangle = \hat{f}(s)\Big|_{s=b_i^*} \tag{6}$$

and the extension from its first derivative

with

$$f_{\psi'_i} \triangleq \left\langle f, \psi'_i \right\rangle = r_i^* \frac{\partial \hat{f}(s)}{\partial s}\bigg|_{s=b_i^*}, \qquad \mathbf{e}_{\overline{\boldsymbol{\psi}}} = \begin{pmatrix} \tilde{\mathbf{f}}_{\psi} - \mathbf{f}_{\psi} \\ \tilde{\mathbf{f}}_{\psi'} - \mathbf{f}_{\psi'} \end{pmatrix}. \tag{7}$$

It should be noted that for a computational convenience the stair case signal (see function `t1` in the Appendix) is modeled rather than the delayed step.

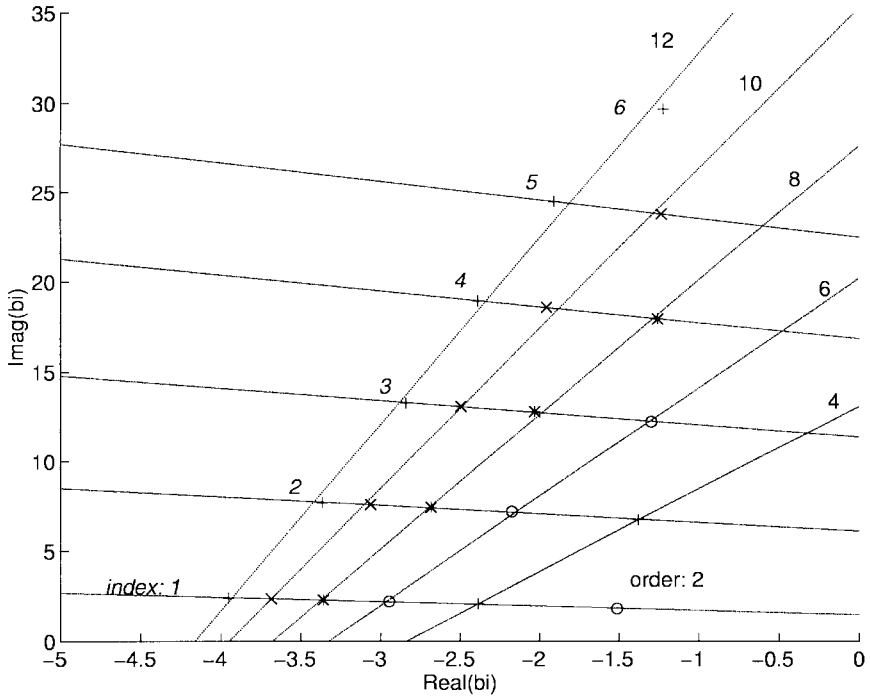

Fig. 1. Upper left-quarter s plane.

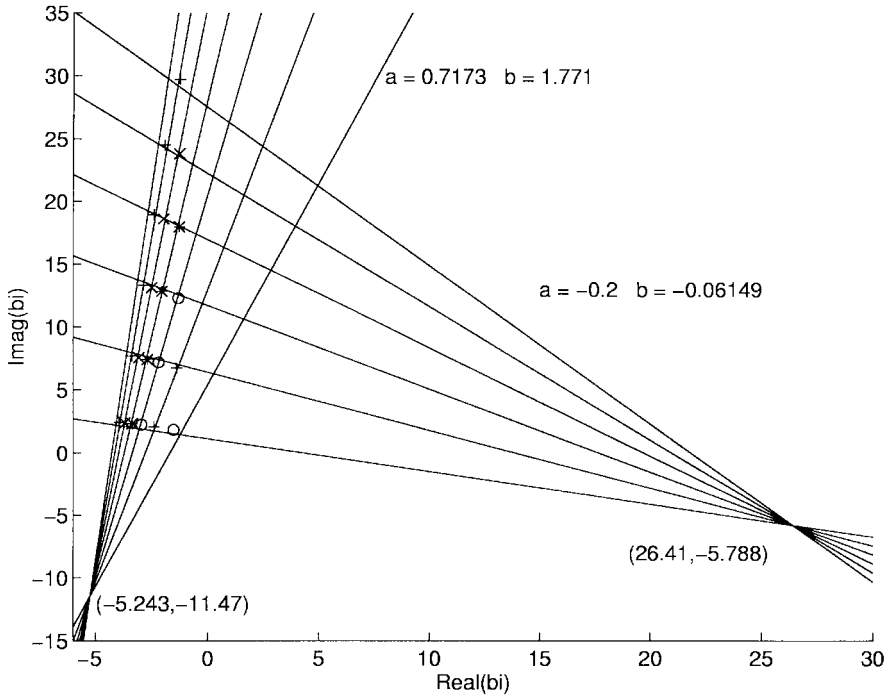

Fig. 2. Line pencils for even orders.

Constraints: The ability of the method to manage linear equality constraints will be greatly appreciated here in order to preserve the low-pass filter behavior of the required model. The initial value of the modeled signal is settled by the following equations:

$$\begin{cases} \displaystyle\sum_{i=1}^{n} r_i \psi_i(0) = 1, & \text{thus } \mathbf{L} = (1, \cdots, 1) \text{ and } y = 1 \\ \overline{\mathbf{L}} = \left(\overbrace{1, \cdots, 1}^{n}, \overbrace{0, \cdots, 0}^{n} \right). \end{cases}$$

In this modeling application, when a great care of the frequency behavior at some prescribed frequencies is wanted, extra constraint equations can be added. Preserving values of the Fourier transform at some prescribed frequencies yields two linear equations for each frequency point.

Starting Procedure: A reliable set of starting poles is of high practical interest. For investigation purposes, basing the selection of initial parameter values on the results of the previous lower order

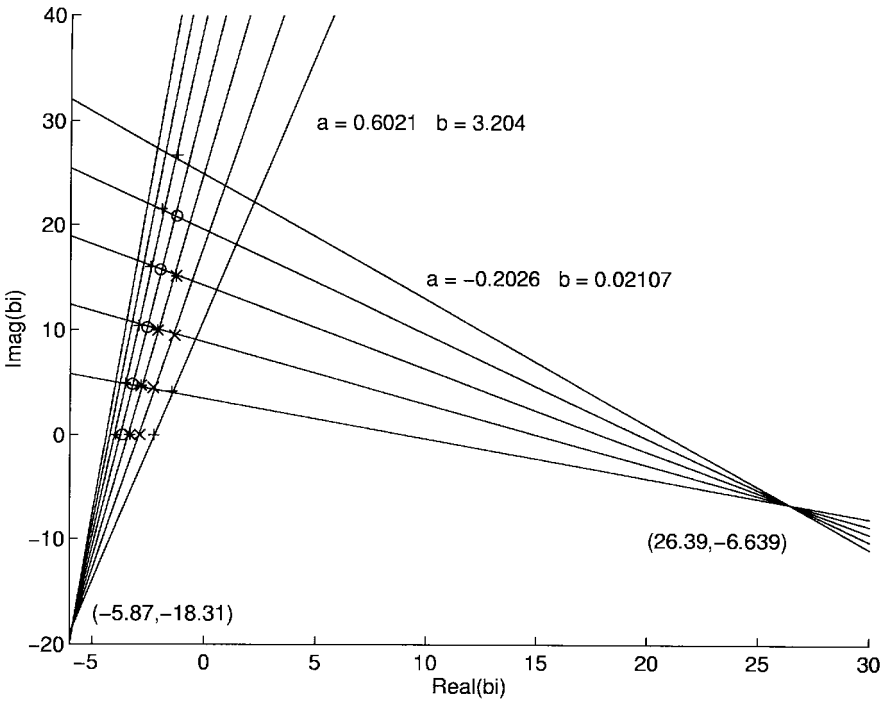

Fig. 3. Line pencils for odd orders.

TABLE I
LINE PENCILS FOR EVEN ORDERS AND ODD ORDERS

	Even orders				Odd orders			
	Point		Slopes		Point		Slopes	
	x_0	y_0	a	b	x_0	y_0	a	b
vs order	-5.243	-11.47	0.7173	1.7771	-5.87	-18.31	0.6021	3.204
vs index	26.41	-5.788	-0.2	-0.06149	26.39	-6.639	-0.2026	0.0207

approximation is a common practice [8]. Here, such an approach must be rejected and an automatic boot procedure is proposed.

Another way would be the initialization with a suboptimal approximation. The known nonoptimal classical Padé approximation could be expected as a worthwhile starting point. Unfortunately, the delay time Padé approximation, rather good for large values of time, is rather poor in the L^2 sense. Moreover, when optimizing, it is experimentally seen that this starting pole map leads invariably to a double pole for orders greater than six.

As was mentioned before, this difficulty has been previously overcome by some authors. However, for the particular pure delay time signal, all the burden of merging poles management (detection, extension of the basis of functions, and extension of the extended basis) will be avoided due to another starting map.

The proposed boot procedure relies on simple geometrical considerations. By a glimpse at the pole positions of optimal models for various orders ($n = 2 \cdots 12$), in Fig. 1, it seems that for each of these even orders, the optimal poles are quasi-aligned. Furthermore, adjoining the poles of a fix index, but with increasing orders, also yields quasi-straight lines. From these two simple remarks the linear regressions which fit the poles by simple straight lines have been systematically computed. It appears that the optimal poles are not so far from the intersection point of two line pencils (Fig. 2). In order to get the slopes σ of the lines of each pencil, the analysis has shown that the variations of the imaginary part of the poles, depending upon the order or upon the index, is quasi-quadratic. Thus, the director coefficients of the straight lines vary quasi-linearly. The pencils are easily built from the two crossing points and from the variation of the slopes σ in parameter p (order or index) as $\sigma = ap + b$. For odd orders (Fig. 3), these three mathematical curiosities are also found. Crossing pencil points and director coefficients are shown in Table I.

Remark 2: From a close inspection of Fig. 1, it can be seen that the optimal poles are not exactly superposed to the pencils' crossing points, nevertheless, these intersections will be used as starting poles for the optimization process.

IV. RESULTS

Before describing the whole optimization process, the first step will be an evaluation of the quality of the proposed starting poles. This set, merely obtained as line intersections is associated with the optimal constrained residues $\mathbf{r} = \mathbf{M}(\mathbf{f}_{\Phi} + \boldsymbol{\chi})$. In Fig. 4 it is seen that the new approximate time delay model (order 11), directly obtained from the line pencils' intersections (without optimization) is better than the same order Padé approximation and very close to the optimal solution. In Fig. 5 it is seen that for odd or even orders

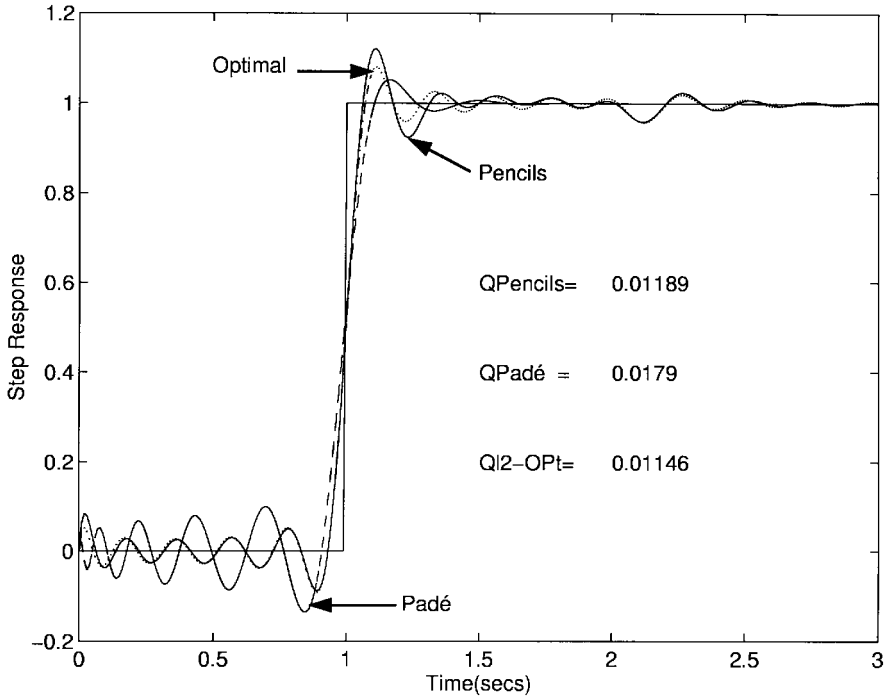

Fig. 4. Order–11 step responses.

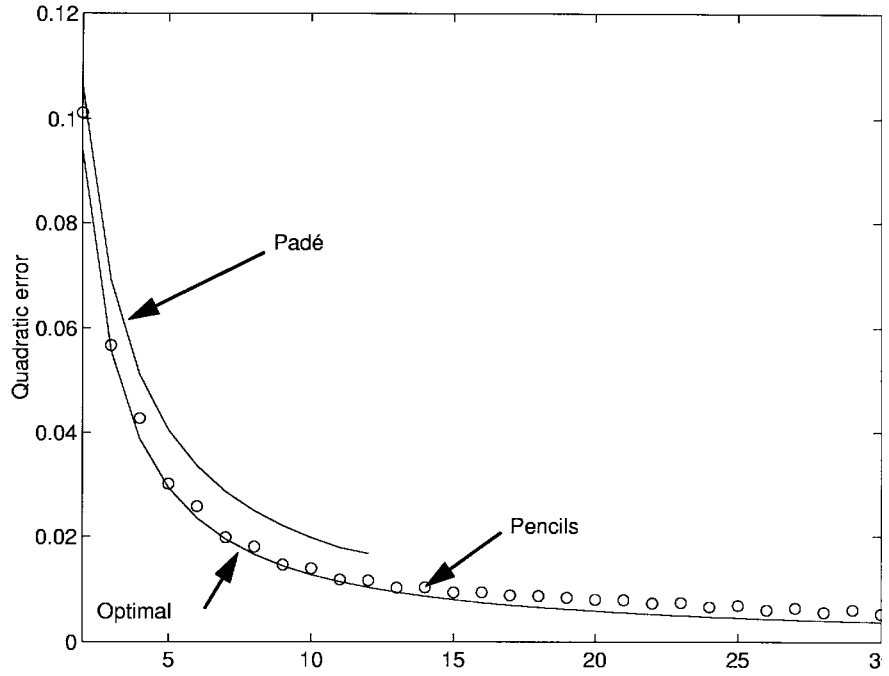

Fig. 5. L^2-errors comparison.

ranging from 2 to 30, the proposed starting solution is more accurate than the Padé approximation. Beyond order 12, the Padé polynomial quotient is of course no longer available. For these higher orders, exponential models arising from the Padé development techniques using state–space representations and reduction methods could be used (see [9] and [10]). They have not been implemented here for comparison purposes, thus, in Fig. 5 the Padé error is not drawn beyond order 12.

The whole optimization process has also been fully implemented and tested in a standard numerical environment, in respect with the IEEE 484 floating point numbers representation (Matlab). It can be seen (Appendix and Fig. 7) that managing the possible instability and monitoring the decrease of the criterion are sufficient to quickly yield high-quality solutions.

Higher order models up to 200 (Fig. 6) can be obtained without software numerical complaint. Beyond that order, the matrix $\overline{\Psi}$ becomes ill conditioned and inversion is inaccurate. It should be noted that, for a similar order, the orthogonalization process complains for a division by zero.

With this really compact algorithm, Fig. 7 shows that two or three iterations are enough for drastically reducing the quadratic error. In Fig. 5 it can be seen that the recommended starting poles are

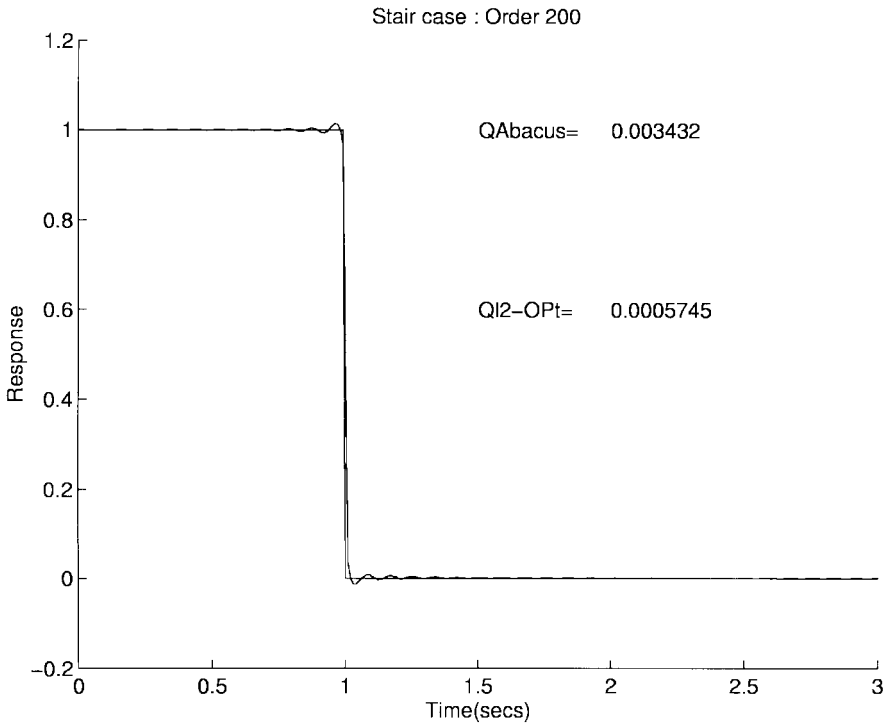

Fig. 6. Order-200 stair case response.

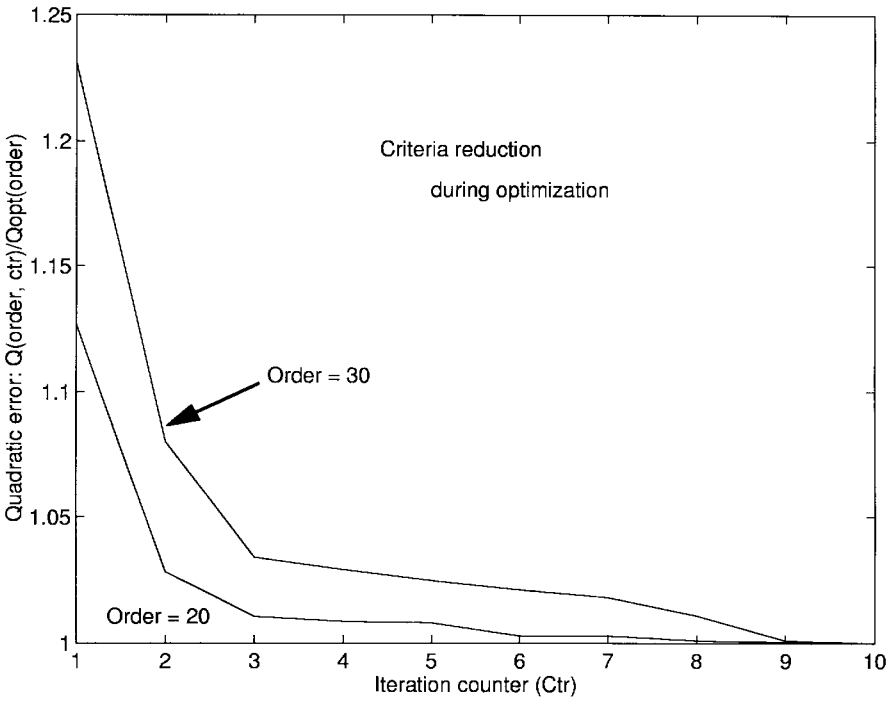

Fig. 7. Criteria reduction factor.

quasi-optimal. However, the exact optimal pole mapping is given in Fig. 8.

V. Conclusion

For engineering purposes one looked for a convenient procedure to get an accurate rational model of the pure time delay (orders $2\cdots10$). The usual Padé approximation, thought as a starting point for the L^2 Gauss–Newton optimization procedure is not successful. It has been seen that for the delayed step during the optimization process, the migration of the poles from the Padé position toward the optimal placement, leads some of them to go through double poles, resulting in serious difficulties.

The burden of managing the poles merging and separation has been avoided with the proposed starting point. Furthermore, it has been shown that, without any optimization iteration, this starting pole map is better than the Padé approximation, largely beyond all usual orders. The initial poles, easily located by simple intersections of two line pencils, are so close to the optimal solution that the

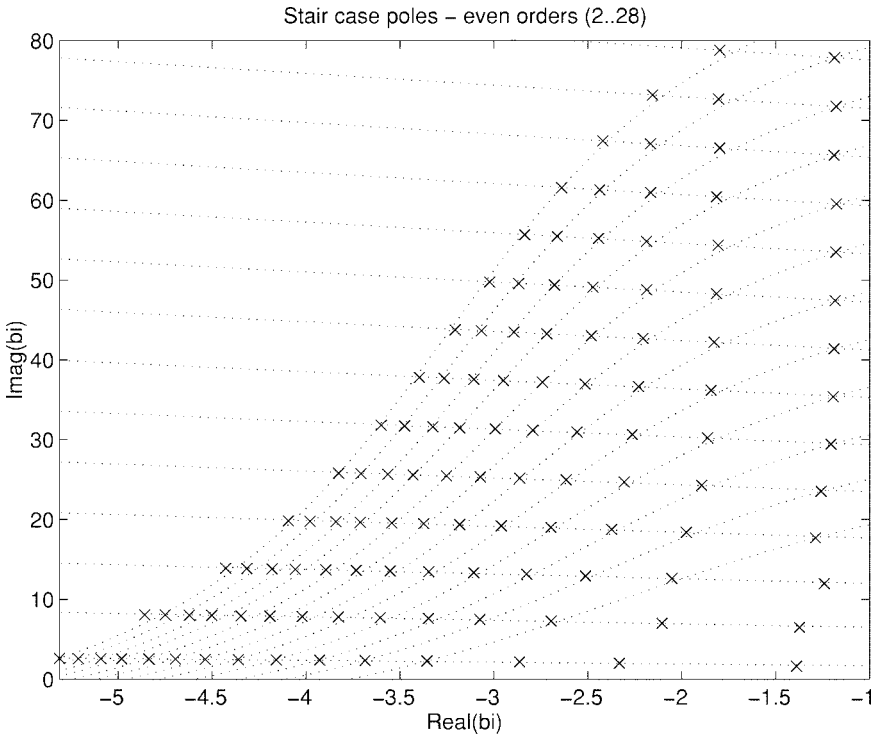

Fig. 8. Optimal poles position.

algorithm immediately locks to the efficient Gauss–Newton iterations. The optimal residues are computed by a compact procedure.

The best (L^2-sense) rational model is obtained after two or three iterations. For numerical robustness validation in the Matlab environment extremely high (up to order 200) optimal time-delay models have been built with the proposed procedure.

The constraining ability has been used here to yield a low-pass filter model. Extra equations could be set to exactly preserve values of the Fourier transform at some prescribed frequencies.

APPENDIX

A. Functions

L^2-Optimal Residues (Poles b_i, Energy E, Constraint Matrices L, y):

```
function [OptRes, Q, MPsi, fPsi, dfPsi] ...
   = L2OptRes(b, E, L, y);
n = length(b);
Mbi = b * ones(1, n);
MPsi = (1)./(Mbi + Mbi');
[fPsi, dfPsi] = tl(conj(b));
M = inv(conj(MPsi'));
Ki = L'*inv(L * M * L') * (y − L * M * fPsi);
OptRes = M * (fPsi + Ki);
Q = real(E − OptRes'*(fPsi − Ki));
```

Gauss–Newton Corrective Term:

```
function [Delta, MPsiBar] ...
   = GNDelta(r, b, MPsi, fPsi, dfPsi, LBar, yBar)
n = length(b);
%Extended basis
MPsi12 = MPsi. * MPsi;
MPsiBar = [MPsi, −MPsi12 * diag(conj(r)) ...
−diag(r) * MPsi12', 2 * diag(r) * MPsi. ...
  MPsi12 * diag(conj(r))];
MBar = inv(conj(MPsiBar'));
%Error projection
fPsiBar = [fPsi; conj(r). * dfPsi];
fTildePsiBar = [conj(MPsiBar(1 : n, 1 : n)') * r; ...
  conj(MPsiBar(1 : n, n + 1 : 2 * n)') * r];
ePsiBar = fTildePsiBar − fPsiBar;
KiBar = LBar'*inv(LBar * MBar * LBar') ...
  *(yBar + LBar * MBar * ePsiBar);
Delta = MBar * (−ePsiBar + KiBar);
```

Stair Case Laplace Transform and Its 1st Derivative:

```
function [F, DF] = tl(val);
c = exp(−val);
F = (1 − c)./val;
DF = (c. * val − (1 − c))./val.^2;
```

B. Main Script: Iterative Procedure

```
%Initial position of the poles
  %(intersection of pencils of lines) :
n = input('even or odd order (1 < n < 40)? =')
p = StepStrt(n);      %to be written from Table I
b = −p
```

```
E = 1.0; %Energy of the model signal
Ctrmax = 10
factdelta = 1; % step factor < 1
erdelta = 0.000001;
if ~all(real(b) > 0) disp('UNSTABLE !!!'); break;
end %if;
Ctr = 0;
Completed = Ctr == Ctrmax;
[r, Q, MPsi, fPsi, dfPsi] = L2OptRes(b, 1.0, ones(1, n), 1);
while ~Completed
[Delta, MPsiBar] = GNDelta(r, b, MPsi, fPsi, dfPsi,
  [ones(1,n),zeros(1,n)], 0);
deltab = Delta(n + 1 : 2 * n);
b1 = b + factdelta * deltab;
delta = deltab;
while ~all(real(b1) > 0)
disp('UNSTABLE - delta must be decreased')
delta = 0.1 * delta; b1 = b + delta;
end; %while
[r, Q1, MPsi, fPsi, dfPsi]
  = L2OptRes(b1, 1.0, ones(1, n), 1);
% Q1 should be smaller
while Q1 >= Q
delta = 0.1 * delta; b1 = b + delta;
[r, Q1, MPsi, fPsi, dfPsi]
  = L2OptRes(b1, 1.0, ones(1, n), 1);
end; %while
Q = Q1;
b = b1;
ndelta = norm(deltab);
fprintf('Ctr = %3i \tpreviousQ %.16e\tndelta
  = %e\n', Ctr, Q/E, ndelta);
%Completed ?
Ctr = Ctr + 1;
Completed = (ndelta < erdelta) | (Ctr >= Ctrmax);
end %while
t = 0 : 0.01 : 3;

yCreOpt = conj(r') * exp(-b * t);
plot(t, real(yCreOpt)).
```

REFERENCES

[1] H. Padé, "Sur la distribution des réduites anormales d'une fonction," *C. R. Acad. Sci.,* pp. 102–104, 1899.

[2] D. W. Kammler and R. J. McGlinn, "A bibliography for approximation with exponential sums," *J. Comput. Appl. Math.,* vol. 4, no. 2, pp. 167–173, 1978.

[3] L. C. Calvez, P. Vilbé, and P. Glouannec, "Orthonormal and nonorthonormal least-squares approximation of a function subject to linear equality constraints," *IEEE Trans. Circuits Syst.,* vol. CAS-32, pp. 851–853, Aug. 1985.

[4] P. Vilbé and L. C. Calvez, "Constrained ℓ_2 suboptimal model reduction," *Electron. Lett.,* vol. 23, no. 25, pp. 1340–1342, Dec. 1987.

[5] P. R. Aigrain and E. M. Williams, "Synthesis of n-reactance networks for desired transient response," *J. Appl. Phys.,* vol. 20, pp. 596–600, 1949.

[6] R. N. Mc Donough and W. H. Huggins, "Best least-squares representation of signals by exponentials," *IEEE Trans. Autom. Contr.,* vol. AC-13, pp. 408–412, 1968.

[7] R. K. Harman, "Closed-form L^2 exponential approximation," Ph.D. dissertation, Queen's Univ., Kingston, Ont., Canada, 1972.

[8] R. K. Harman and F. W. Fairman, "Exponential approximation via a closed-form Gauss–Newton method," *IEEE Trans. Circuit Theory,* vol. CT-20, pp. 361–369, 1973.

[9] J. Lam, "Model reduction of delay systems," Ph.D. dissertation, Darwin College, 1988.

[10] K. Glover, J. Lam, and J. R. Partington, "Rational approximation of a class of infinite-dimensional systems—Part II: Optimal convergence rates of L_∞ approximants," *Math. Control, Signals, Syst.,* vol. 4, pp. 233–246, 1991.

A System Theoretic Approach to the Stability of Passively *Q*-Switched Lasers

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***Abstract*—The rate equations of passively *Q*-switched lasers are considered. These equations represent the dynamics of the photon number in the laser cavity, the population inversion of the laser, and the ground-state population of the saturable absorber when excited by an input (pumping rate). In this note it is shown that for bounded inputs the laser output is bounded. Furthermore, it is shown that when the input is switched off, the laser output converges to zero asymptotically. These stability results show that passively *Q*-switched lasers can operate safely.**

***Index Terms*—Lyapunov technique, passively *Q*-switched lasers, rate equations, stability.**

I. INTRODUCTION

Q-switched lasers generate intense (giant) pulses of short duration of a few tens of nanoseconds. The mechanism for generating such pulses is to initially keep the laser cavity from oscillating by increasing the cavity losses to high values or removing the cavity feedback, while letting the laser pumping build up a large population inversion. Then, after a large population inversion is accumulated, the cavity feedback is restored by means of some rapid modulation. This process results in an intense pulse of short duration. Intense pulses generated by Q-switched lasers are important in many applications, such as laser radars, tunable laser radars, cutting, drilling, and welding, and scientific experiments in nonlinear optics. Due to their importance, Q-switched lasers have been studied by researchers extensively (see, e.g., [1]–[5] and the references therein). There are different types

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