Pergamon

PII: S0016–0032(96)00138–X

J. Franklin Inst. Vol. 335B, No. 8, pp. 1507–1518, 1998

0016–0032/98 $19.00+0.00

Balanced Realization using an Orthogonalization Procedure and Modular Polynomial Arithmetic

by S. AZOU, P. VILBE *and* L. C. CALVEZ

LEST/UBO-UMR CNRS 6616, BP 809, 29285 Brest Cedex, France

(Received 27 July 1996; accepted 6 December 1996)

ABSTRACT: *A new procedure for deriving a balanced realization of continuous time or discrete time block-factorized transfer function is proposed. This work is based on orthogonalization processes of input maps through use of Routh/Aström tables and modular polynomial arithmetic.* © 1998 The Franklin Institute. Published by Elsevier Science Ltd.

I. Introduction

Balanced realizations of linear dynamical systems have become an even more indispensable tool in the model reduction domain (**1–3**) or for the synthesis of minimum roundoff noise digital filters (**4**). For this purpose, many methods have been proposed for achieving a balanced realization from a given transfer function matrix (TFM). Starting with a known TFM, various methods avoiding the numerical solution of the Lyapunov equations have been developed: some of them deal with the known poles of the system (factorized form, restricted or not to simple and real poles) (**5–8**), other methods apply to the rational form of the Laplace or z TFM (**9–14**).

Nevertheless, this great diversity of various methods, each of them dealing with each specific form, can be seen as the actual weakness of balancing algorithms impeding a larger use in various engineering domains. For solving those various cases, the present paper describes a unified way which relies on orthogonal input maps. The Input/Output (I/O) maps have been introduced first by Burns and Fairman when the poles of the system are known (**8**). However, managing complex and multiple poles gives rise to involved balancing algorithms.

In this paper, we merge the approaches suggested in (**8**) or (**10–13**) in a unified way with the help of I/O maps, and extend the orthogonalization procedure to solve the frequent engineering case of block-factorized Laplace or z TFM. This work is based on orthogonalization processes of input maps through use of Routh/Aström tables (**17**) and modular polynomial arithmetic. The background concerning Input/Output maps is first presented in section 2. Use of Routh/Aström tables for the orthogonalization of input maps and extension to discrete or continuous time block-factorized TFM is described in section 3. Section 4 is then devoted to the computation of a minimal balanced state space realization. Finally, a numerical example is given in section 5.

II. Realizations and Input/Output Maps

As stated by Burns and Fairman in (**8**), I/O maps yield a worthwhile representation of a given TFM in order to build the corresponding balanced state space realization. These maps lead to computationally attractive methods avoiding the solution of the Lyapunov equations. In (**8**), an orthogonal input map is built from the modes of the system and then, the observability Gramian is obtained in a convenient manner which gives the balanced realization from a simple coordinate transformation based on the eigenvalues and eigenvectors. However, dealing with I/O maps in the Laplace or z domain is more convenient for a unified approach (**10–13**).

Starting with the most general form of TFM (block-factorized TFM), this section states the basic properties of I/O maps for continuous or discrete time systems.

Definition 1: For any realization (A, B, C, D) of a continuous time impulse response matrix $H(t)$, the I/O maps $L(t)$, $M(t)$ are defined as:

$$\begin{cases} L(t) = \mathrm{e}^{At}B \\ M(t) = \mathrm{e}^{A^{\$}t}C^{\$} \end{cases} \tag{1}$$

where \$ designs the conjugate-transpose.

Theorem I: $L(t)$ is an input map for the continuous time impulse response matrix $H(t)$ iff:

$$\exists(A, C) \quad H(t) = CL(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}L(t) = AL(t). \tag{2}$$

In the discrete time case we have the following similar results:

Definition 2: For any realization (A, B, C, D) of the discrete time impulse response matrix $H[n]$, the I/O maps $L[n]$, $M[n]$ can be defined as:

$$\begin{cases} L[n] = A^{n-1}B, n \geqslant 1; & L[0] = 0 \\ M[n] = (A^{n-1})^{\$}C^{\$}, n \geqslant 1; & M[0] = 0 \end{cases}. \tag{3}$$

Theorem II: $L[n]$ is an input map for the discrete time impulse response matrix $H[n]$ iff:

$$\exists(A, C) \quad H[n] = CL[n], \quad L[n+2] = AL[n+1]. \tag{4}$$

In the sequel, the input maps will be assumed to have independent rows. A such input map is particularly attractive because the corresponding (controllable) realization is unique.

Furthermore, usefulness of orthogonal input maps has been pointed out for achieving a balanced realization.

Definition 3: Let $L(t)$ be a continuous time input map and $L[n]$ be a discrete time input map (having independent rows) for any continuous/discrete time impulse response matrix. Let W_c, W_o be the controllability/observability Gramians for the unique corresponding realization (A, B, C, D) (the existence follows from the stability of the TFM):

$$W_c = \langle L, L^T \rangle = \int_0^\infty L(t)L^\$(t)\,dt; \quad W_o = \langle M, M^T \rangle = \int_0^\infty M(t)M^\$(t)\,dt \tag{5}$$

(continuous time case)

$$W_c = \langle L, L^T \rangle = \sum_{n=0}^{\infty} L[n]L^\$[n]; \quad W_o = \langle M, M^T \rangle = \sum_{n=0}^{\infty} M[n]M^\$[n] \tag{6}$$

(discrete time case).

The input map L is said to be orthogonal, and then noted $L_\perp$, if the controllability Gramian is diagonal. A realization will be called an input orthogonal realization, and then written $(A_\perp, B_\perp, C_\perp, D_\perp)$, if the corresponding input map is orthogonal.

It follows that the problem of determining a balanced realization of a TFM is reduced to a single eigenvalue–eigenvector problem once an input orthogonal realization and its observability Gramian $W_{o\perp}$ have been determined [the eigenvalues of the controllability/observability Gramians product are the squared second-order modes (**15**)].

In the sequel, interest will focus on TFM given in the following block-factorized form, avoiding the iterative pole finding process or the inaccurate representation of high order polynomial coefficients:

$$H(q) = [h_{ij}(q)]\begin{matrix} i = 1, \ldots, v \\ j = 1, \ldots, u \end{matrix} \quad \text{with} \quad h_{ij}(q) = \prod_{k=1}^{m} \frac{N_{ij,k}(q)}{D_k(q)} \tag{7}$$

with $q = s$ in the continuous time case, $q = z$ in the discrete time case and where $\{D_k(q); k = 1, \ldots, m\}$ is a set of pairwise relatively prime polynomials, of degree $n_1, \ldots, n_m$ respectively.

In the following theorem, initial I/O maps are derived directly from the TFM:

Theorem III: Let the given stable continuous/discrete MIMO system be described by its TFM:

$$H(q) = [h_{ij}(q)]\begin{matrix} i = 1, \ldots, v \\ j = 1, \ldots, u \end{matrix} \quad \text{with} \quad h_{ij}(q) = \prod_{k=1}^{m} \frac{N_{ij,k}(q)}{D_k(q)} = g_{ij} + \sum_{k=1}^{m} \frac{P_{ij,k}(q)}{D_k(q)} \tag{8}$$

where $g_{ij} \in \mathbf{R}$ and $\deg[P_{ij,k}] < \deg[D_k]$.

The derivation of the $\{g_{ij}, P_{ij,k}\}$ can be performed in an efficient manner as it is pointed out in the next section.

1. The following rational matrix is an input map for $H(q)$:

$$L_0(q) = I_u \otimes \boldsymbol{\Psi}(q), \quad \text{with} \quad \boldsymbol{\Psi}(q) = [\boldsymbol{\Psi}_1^T(q), \ldots, \boldsymbol{\Psi}_m^T(q)]^T \tag{9}$$

where

$$\mathbf{\Psi}_k^{\mathrm{T}}(q) = \frac{1}{D_k(q)}[1, \ldots, q^{n_k - 1}], \quad k = 1, \ldots, m; \quad n_k = \deg[D_k].$$

2. The realization (A_0, B_0, C_0, D_0) corresponding to this input map is obtained as

$$\begin{cases} A_0 = I_u \otimes A_c \quad \text{with} \quad A_c = \operatorname{diag}[A_{c_1}, \ldots, A_{c_m}] \\ B_0 = I_u \otimes B_c \quad \text{with} \quad B_c = [B_{c_1}^{\mathrm{T}}, \ldots, B_{c_m}^{\mathrm{T}}]^{\mathrm{T}} \\ C_0 = [C_c^{ij}]\begin{smallmatrix} i = 1, \ldots, v \\ j = 1, \ldots, u \end{smallmatrix} \quad \text{with} \quad C_c^{ij} = [C_{c_1}^{ij}, \ldots, C_{c_m}^{ij}] \\ D_0 = [g_{ij}]\begin{smallmatrix} i = 1, \ldots, v \\ j = 1, \ldots, u \end{smallmatrix} \end{cases} \tag{10}$$

where $(A_{c_k}, B_{c_k}, C_{c_k}^{ij})$ is the controllable realization for the rational fraction $P_{ij,k}(q)/D_k(q)$, with A_{c_k} the bottom companion matrix for D_k, $B_{c_k} = [0, \ldots, 0, 1]^{\mathrm{T}}$ and C_{c_k} elements being the $P_{ij,k}$ coefficients.

3. The corresponding output map is given by the relation:

$$M_0(q) = F_0[I_v \otimes \mathbf{\Psi}(q)], \quad \text{with} \quad F_0 = \begin{bmatrix} F_{11} & \ldots & F_{v1} \\ \vdots & \ddots & \vdots \\ F_{1u} & \ldots & F_{vu} \end{bmatrix} \tag{11}$$

where $F_{ij} = \operatorname{diag}[F_{ij,1}, \ldots, F_{ij,m}]$, $F_{ij,k}$ being the Bezout matrix for the polynomials $(P_{ij,k}, D_k)$.

Proof: It is easily seen that the matrix $L_0(q)$ in Eq. (9) satisfies the relations in Eqns (2) and (4) with the matrices (A_0, C_0) given in Eq. (10). It follows from this observation that (A_0, B_0, C_0, D_0) in Eq. (10) is the realization for L_0 in Eq. (9). It has been shown in (**10**, **11**) that the I/O maps for a bottom companion controllable realization of any scalar transfer function is the Bezout matrix of its numerator and denominator. Then the result in Eq. (11) is immediately derived. ■

Remark 1: As stated in the above theorem, an initial input map can be obtained directly from the TFM using the block companion realization in Eq. (10). The corresponding output map is easily achieved owing to Bezout matrices [Eq. (11)]. However, we must emphasize that in the present paper, Bezoutians are not employed as a criteria for checking the relative primeness of polynomials.

Theorem III extends I/O maps properties to the most general case of block-factorized TFM. These results will be used in the balancing procedure: orthogonalization of input maps is first considered.

III. Input Orthogonal Realization Using Modular Polynomial Arithmetic

It has been shown that orthogonal input maps are highly desirable when balancing transfer function matrices: first results have been obtained by Burns and Fairman when orthogonalizing exponential functions in the time-domain (**8**). In the Laplace domain (resp. in the z domain) it has been pointed out that the classical Routh table (resp. Aström table) gives rise to an orthogonal set of functions (**10**) [resp. (**11**, **17**)]. Interesting results based on these properties have been proposed to derive balanced realizations using I/O maps (**10–13**).

In the sequel, extended results are stated which deal with block-factorized continuous and discrete time TFM [Eq. (7)]. For this synthetic method, the input map is obtained from an extended orthogonal set and the corresponding orthogonalization matrix is computed via modular polynomial arithmetic.

Theorem IV: Let the given stable continuous/discrete time MIMO system be described by its TFM [Eq. (7)].

In the continuous time case, each polynomial $D_k(s)$ gives rise to an orthogonal set extracted from the associated Routh table:

$$D_k(s), \quad k = 1, \ldots, m \rightarrow \{\hat{\varphi}_l^k(s) = \frac{A_l^k(s)}{D_k(s)}; \quad l = 1, \ldots, n_k\} \tag{12}$$

with $\langle \varphi_i^k, \varphi_j^k \rangle = \sigma_i^{k2}\, \delta_{i,j}$, where $\delta_{i,j}$ is the Kronecker delta.

In the discrete time case, each polynomial $D_k(z)$ gives rise to an orthogonal set extracted from the associated Aström table:

$$D_k(z), \quad k = 1, \ldots, m \rightarrow \left\{\varphi_l^k(z) = \frac{A_l^k(z)}{D_k(z)}; \quad l = 1, \ldots, n_k\right\} \tag{13}$$

with $\langle \varphi_i^k, \varphi_j^k \rangle = \sigma_i^{k2}\, \delta_{i,j}$.

Let

$$\mathbf{\Phi}(q) = [\mathbf{\Phi}_1^{\mathrm{T}}(q), \ldots, \mathbf{\Phi}_m^{\mathrm{T}}(q)]^{\mathrm{T}} \tag{14}$$

with

$$\begin{cases} \mathbf{\Phi}_k^{\mathrm{T}}(q) = [\hat{\varphi}_1^k(s), \ldots, \hat{\varphi}_{n_k}^k(s)] \cdot \prod_{l=1}^{k-1} \dfrac{D_l(-s)}{D_l(s)}, & \text{if } q = s \quad (k > 1) \\ \mathbf{\Phi}_k^{\mathrm{T}}(q) = [\varphi_1^k(z), \ldots, \varphi_{n_k}^k(z)] \cdot \prod_{l=1}^{k-1} \dfrac{D_l^*(z)}{D_l(z)}, \quad \text{with} \quad D_l^*(z) = z^{n_k} D_k\left(\dfrac{1}{z}\right), & \text{if } q = z \quad (k > 1) \end{cases}. \tag{15}$$

Then $L_\perp(q) = I_u \otimes \mathbf{\Phi}(q)$ is an orthogonal input map for $H(q)$.

Proof: It follows from (**18**) that the functions of the column vector $\mathbf{\Phi}(q)$ form an orthogonal basis. Using Theorems I and II, it is easily seen that $L_\perp(q) = I_u \otimes \mathbf{\Phi}(q)$ is an input map for $H(q)$. Then it follows that $L_\perp(q)$ is an orthogonal input map. ■

Then an orthogonal realization can be easily derived through use of an orthogonalization matrix.

Theorem V: The following realization $(A_\perp, B_\perp, C_\perp, D_\perp)$ of $H(q)$ is input orthogonal and controllable:

$$\begin{cases} A_\perp = (I_u \otimes C)A_0(I_u \otimes C^{-1}) \\ B_\perp = (I_u \otimes C)B_0 \\ C_\perp = C_0(I_u \otimes C^{-1}) \\ D_\perp = D_0 \end{cases}. \tag{16}$$

The controllability/observability Gramians are:

$$\begin{cases} W_{c\perp} = I_{(n\times u)} \\ W_{o\perp} = F_\perp F_\perp^{\mathrm{T}} \\ \text{with} \quad F_\perp = (I_u \otimes C^{-\mathrm{T}})F_0(I_v \otimes C^{-1}) \end{cases} \tag{17}$$

where the orthogonalization matrix C is defined by:

$$L_\perp(q) = (I_u \otimes C)L_0(q). \tag{18}$$

$L_0(q)$ being the input map for the initial realization (A_0, B_0, C_0, D_0) defined in Theorem I.

Proof: It is easily seen that each function of the orthogonal vector $\mathbf{\Phi}(q)$ defined in Eqns (14) and (15) can be expressed as a linear combination of the functions of the initial column vector $\mathbf{\Psi}(q)$ in Eq. (9): $\mathbf{\Phi}(q) = C\mathbf{\Psi}(q)$ (C being the orthogonalization matrix). Then, it enables the orthogonal input map $L_\perp(q)$ to be written as $L_\perp(q) = (I_u \otimes C)L_0(q)$, with $L_0(q) = I_u \otimes \mathbf{\Psi}(q)$ the initial input map. The input orthogonal realization $(A_\perp, B_\perp, C_\perp, D_\perp)$ is then obtained from (A_0, B_0, C_0, D_0) through the similarity transformation $(I_u \otimes C^{-1})$. According to the foregoing it follows that the corresponding controllability Gramian is $W_{c\perp} = I_{(n\times u)}$. The output map of $(A_\perp, B_\perp, C_\perp, D_\perp)$ is given by $M_\perp = (I_u \otimes C^{-\mathrm{T}})M_0$ with M_0 in Eq. (11), which gives rise to the final expression of the observability Gramian $W_{o\perp}$ in Eq. (17). ■

It is shown in the sequel that the orthogonalization matrix C can be derived in an efficient manner using modular polynomial arithmetic.

In the elementary case where the denominator $D(q)$ is common to the elements $h_{ij}(q)$ of the TFM, the rows of C are readily copied from the rows of the Routh/Aström table built to verify the stability of $D(q)$. The matrix C is then lower triangular.

Let us consider now the more general case where the least common denominator of $H(q)$ has the factorized form $D(q) = \Pi_{k=1}^{m} D_k(q) (m > 1)$. It is easily seen that the $n_1 = \deg[D_1]$ first rows of C are derived proceeding the same way. The other rational (strictly proper) functions of the orthogonal vector $\mathbf{\Phi}(q)$ have a factorized form [Eqns (14) and (15)]. To compute the corresponding rows in the matrix C, rewrite each of them using a partial fraction decomposition:

$$\frac{A_j^k(q)}{D_k(q)} \prod_{l=1}^{k-1} \frac{\bar{D}_l(q)}{D_l(q)} = \sum_{l=1}^{k} \frac{P_{j,l}^k(q)}{D_l(q)}, \quad k = 2, \ldots, m, \quad j = 1, \ldots, n_k$$

where $\bar{D}_l(q) = D_l(-s)$ in the continuous time case and $\bar{D}_l(q) = z^{n_l} D_l(1/z)$ in the discrete time case. The $(\Sigma_{t=1}^{k-1} n_t + j)$th row of C is then immediately obtained by copying the coefficients of the polynomials $P_{j,l}^{k}(q)$.

It has been shown in (**19**) that use of modular polynomial arithmetic is worthwhile to do such rational fractions decomposition since it gives rise to fast algorithms. Then the orthogonalization matrix can be efficiently determined since all the computations involved to decompose the factorized functions in Eqns (14) and (15) are done using low degree polynomials.

Some results concerning modular polynomial arithmetic can be found in the Appendix.

IV. Balanced Realization

As a direct application of the previous results, a method is proposed to achieve balanced realizations of MIMO systems known by their TFM [Eq. (7)]. Due to the orthogonalization procedures described above, the technique applies as well for continuous or discrete time systems.

The steps involved in the algorithm are summarized in the following:

1. Compute the numbers g_{ij} and the polynomials $P_{ij,k}(q)$ in Eq. (8) using the partial fraction decomposition algorithm described in the Appendix.
2. An initial controllable realization (A_0, B_0, C_0, D_0) with I/O maps $L_0(q)$, $M_0(q)$ is given by Eqns (9)–(11).
3. Form an orthogonal input map $L_\perp(q)$ as described in Theorem IV. The functions extracted from the Routh/Aström tables will be normalized here.
4. Compute the orthonormalization matrix C such that $L_\perp(q) = (I_u \otimes C)L_0(q)$ with the aid of the partial fraction decomposition algorithm.
5. Compute the input orthogonal realization $(A_\perp, B_\perp, C_\perp, D_\perp)$ which is deducted from (A_0, B_0, C_0, D_0) by the similarity transformation $(I_u \otimes C^{-1})$. Note that the computation of the inverse matrix C^{-1} can be done in an efficient manner using the particular staircase form of C. The corresponding controllability/observability Gramians $W_{\mathrm{c}\perp}$, $W_{\mathrm{o}\perp}$ are given by Eq. (17).
6. Eigenvalue–eigenvector decompose $W_{\mathrm{o}\perp}$:

$$Q^{\mathrm{T}} W_{\mathrm{o}\perp} Q = \Sigma'^2, \quad \text{with} \quad \Sigma' = \mathrm{diag}\,[\sigma_1, \ldots, \sigma_r, 0, \ldots, 0] = \mathrm{diag}\,[\Sigma, \mathbf{0}] \tag{19}$$

   where the σ_i, $i = 1, \ldots, r$ denote the second-order modes and r the Mac-Millan degree of $H(q)$.

   Note that with the proposed procedure no minimality of the transfer functions h_{ij} in Eq. (7) is required. Hence, due to the numerical inaccuracies, some σ_i corresponding to the unobservable part (in the case of a nonminimal input orthogonal realization) may not be exact zeros. Then, r will be an estimation of the Mac-Millan degree of the TFM.
7. Use the Q matrix as a similarity transformation to obtain the following input orthogonal realization $(A'_\perp, B'_\perp, C'_\perp, D'_\perp)$ from $(A_\perp, B_\perp, C_\perp, D_\perp)$, the Gramians being $W'_{\mathrm{c}\perp} = I_{(n\times n)}$, $W'_{\mathrm{o}\perp} = \Sigma'^2$.
8. Reduce $(A'_\perp, B'_\perp, C'_\perp, D'_\perp)$ to an r-order irreducible form $(A''_\perp, B''_\perp, C''_\perp, D''_\perp)$ by eliminating the unobservable part.

9. Finally, use of the similarity transformation $\Sigma^{-1/2}$ starting from $(A''_{\perp}, B''_{\perp}, C''_{\perp}, D''_{\perp})$ leads to a minimal balanced realization (A_b, B_b, C_b, D_b) with controllability/observability Gramians equal to Σ.

V. Example

The present method is particularly convenient for the design of analog circuits where the cascade of elementary circuits yields high order block-factorized transfer functions. In the discrete time domain, low order models and robust realizations are also welcome: the following example, appearing in the design of digital filters, has been proposed by Mullis and Roberts (**20**). For the redactional purpose the order of the Butterworth low-pass filter has been limited to a low value (6).

$$H(z) = \left(\frac{9.8\times 10^{-4}(z+1)^2}{z^2-1.9641z+0.96802}\right)\left(\frac{9.45\times 10^{-4}(z+1)^2}{z^2-1.9112z+0.91498}\right)\left(\frac{9.325\times 10^{-4}(z+1)^2}{z^2-1.8819z+0.88563}\right).$$

The input map of the initial block controllable realization is given by:

$$L_0(z) = \mathbf{\Psi}(z) = \begin{bmatrix} \dfrac{1}{z^2-1.9641z+0.96802}[1 \quad z]^{\mathrm{T}} \\ \dfrac{1}{z^2-1.9112z+0.91498}[1 \quad z]^{\mathrm{T}} \\ \dfrac{1}{z^2-1.8819z+0.88563}[1 \quad z]^{\mathrm{T}} \end{bmatrix}.$$

The output map being $M_0(z) = F_0\mathbf{\Psi}(z)$, with

$$F_0 = \begin{bmatrix} 0.027283 & -0.025191 & 0 & 0 & 0 & 0 \\ -0.025191 & 0.022929 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.16859 & 0.17646 & 0 & 0 \\ 0 & 0 & 0.17646 & -0.18433 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.13511 & -0.14775 \\ 0 & 0 & 0 & 0 & -0.14775 & 0.16140 \end{bmatrix}.$$

Then the required orthogonal input map is easily built from the Aström tables of each order-2 denominator.

The two functions $\varphi_1(z)$, $\varphi_2(z)$, elements of vector $\mathbf{\Phi}_1(z)$ derived from the first denominator, are readily copied from the standard Aström table:

$$\begin{array}{cccc} 1 & -1.9641 & 0.96802 & \\ 0.062937 & -0.062811 & & \alpha_2 = 0.96802 \\ 2.5047e-0.4 & & & \alpha_1 = -0.997998 \end{array}.$$

Thus $\varphi_1(z) = (2.5047\times 10^{-4})/(z^2-1.9641z+0.96802)$, $\varphi_2(z) = (0.0629z-0.0628)/(z^2-1.9641z+0.96802)$ are the two first orthogonal functions. Proceeding the same way for $\mathbf{\Phi}_2(z)$, $\mathbf{\Phi}_3(z)$, normalizing the scalar products and then using Eqns (14) and (15), gives rise to the orthonormal input map:

$$L_\perp(z) = \begin{bmatrix} \dfrac{1}{z^2-1.9641z+0.96802}\begin{bmatrix} 0.015826 \\ 0.25087z-0.25037 \end{bmatrix} \\ \dfrac{1}{z^2-1.9112z+0.91498}\,\dfrac{0.96802z^2-1.9641z+1}{z^2-1.9641z+0.96802}\begin{bmatrix} 0.025340 \\ 0.40350z-0.40270 \end{bmatrix} \\ \dfrac{1}{z^2-1.8819z+0.88563}\,\dfrac{0.91498z^2-1.9112z+1}{z^2-1.9112z+0.91498}\,\dfrac{0.96802z^2-1.9641z+1}{z^2-1.9641z+0.96802} \\ \times\begin{bmatrix} 0.029195 \\ 0.46439z-0.46347 \end{bmatrix} \end{bmatrix}.$$

Writing each function of $L_\perp(z)$ as a linear combination of $\mathbf{\Psi}(z)$ gives the staircase orthogonalization matrix $L_\perp(z) = C\mathbf{\Psi}(z)$ with

$$C = \begin{bmatrix} 0.015826 & 0 & 0 & 0 & 0 & 0 \\ -0.25037 & 0.25087 & 0 & 0 & 0 & 0 \\ -0.034768 & 4.8676\times10^{-3} & 0.05904 & -4.8676\times10^{-3} & 0 & 0 \\ 0.4775 & -0.47875 & -0.86734 & 0.86934 & 0 & 0 \\ 0.062672 & -0.015145 & -0.35554 & 7.2348\times10^{-3} & 0.31976 & 7.9098\times10^{-3} \\ -0.76172 & 0.76416 & 5.5388 & -5.5502 & -5.1875 & 5.1974 \end{bmatrix}.$$

Use of the similarity transformation C^{-1} then yields an input orthogonal realization $(A_\perp, B_\perp, C_\perp, D_\perp)$, with the following observability Gramian [Eq. (17)]:

$$W_{o\perp} = \begin{bmatrix} 0.6045 & 0.021287 & 0.23084 & 0.17539 & 0.014685 & 0.016078 \\ 0.021287 & 0.30363 & -0.23214 & -0.045291 & -0.039052 & -0.033372 \\ 0.23084 & -0.23214 & 0.49364 & 0.13303 & 0.086212 & 0.073233 \\ 0.17539 & -0.045291 & 0.13303 & 0.068885 & 0.014507 & 0.013637 \\ 0.014685 & -0.039052 & 0.086212 & 0.014507 & 0.017723 & 0.014638 \\ 0.016078 & -0.033372 & 0.073233 & 0.013637 & 0.014638 & 0.012147 \end{bmatrix}.$$

The second-order modes σ_i are then computed via the standard eigenvalue/eigenvector decomposition of $W_{o\perp} : Q^{\mathrm{T}} W_{o\perp} Q = \Sigma^2$, with $\Sigma = \mathrm{diag}\{\sigma_i\} = \mathrm{diag}$ [0.9469, 0.7003, 0.32622, 0.083308, 0.011126, 6.3808×10^{-4}].

This minimal balanced realization is then computed using the similarity transformation $\Sigma^{-1/2}$:

$$A_{\mathrm{b}} = 10^{-3} \times \begin{bmatrix} 997.17 & -32.286 & 5.9946 & -6.1345 & 2.3016 & 0.61143 \\ 32.286 & 988.33 & 40.77 & -12.22 & 6.0225 & 1.5399 \\ 5.9946 & -40.77 & 973.7 & 44.043 & -13.14 & -3.5866 \\ 6.1345 & -12.22 & -44.043 & 954.81 & 43.911 & 9.6995 \\ 2.3016 & -6.0225 & -13.14 & -43.911 & 933.61 & -37.423 \\ -0.61143 & 1.5399 & 3.5866 & 9.6995 & 37.423 & 909.57 \end{bmatrix}$$

$$B_{\mathrm{b}} = [0.067829 \quad -0.12126 \quad -0.12478 \quad -0.081022 \quad -0.034352 \quad 8.9751 \times 10^{-3}]$$

$$C_{\mathrm{b}} = [0.067829 \quad 0.12126 \quad -0.12478 \quad 0.081022 \quad -0.034352 \quad -8.9751 \times 10^{-3}]$$

$$D_{\mathrm{b}} = 8.6359 \times 10^{-10}.$$

VI. Conclusion

An efficient technique for deriving an input orthogonal realization from a given block-factorized transfer function matrix has been proposed. Extended orthogonal sets in the Laplace or z-domain are easily obtained directly from the coefficients of the elementary blocks via Routh/Aström tables. Use of an algorithm based on modular polynomial arithmetic then yields an orthogonalization matrix in a convenient manner since computations with inaccurate high degree polynomials are avoided. An application of the method for achieving a minimal balanced state space realization has been shown. Hence, no resolution of Lyapunov equations is needed and it as well applies to factorized or not, continuous or discrete time systems.

References

(1) B. C. Moore, "Principal components analysis in linear systems: controllability, observability, and model reduction", *IEEE Trans. Automat. Contr.*, Vol. AC-26, pp. 17–32, 1981.

(2) E. I. Verriest and T. Kailath, "On generalized balanced realizations", *IEEE Trans. Automat. Contr.*, Vol. AC-28, pp. 833–844, 1983.

(3) L. M. Silverman and M. Bettayeb, "Optimal approximation of linear systems", in "Proc. JACC", San Francisco, CA, 1980.

(4) C. T. Mullis and R. A. Roberts, "Synthesis of minimum roundoff noise fixed point digital filters", *IEEE Trans. Circ. & Syst.*, Vol. CAS-23, pp. 551–562, 1976.

(5) F. W. Fairman, S. S. Mahil and J. A. De Abreu, "Balanced realization algorithm for scalar continuous-time systems having simple poles", *Int. J. Sci.*, Vol. 15, pp. 685–694, 1984.

(6) J. A. De Abreu and F. W. Fairman, "Balanced realization of orthogonally symmetric transfer function matrices", *IEEE Trans. Circ. & Syst.*, Vol. CAS-34(9), pp. 997–1009, Sept. 1987.

(7) B. S. Lee and F. W. Fairman, "Direct minimal realization from impulse response matrices using I/O map decompositon", *Siam J. Contr. & Opt.*, Vol. 25(1), pp. 1–17, Jan. 1987.

(8) P. D. Burns and F. W. Fairman, "Representation and realization of stable systems using input maps", *IEE Proc.*, Vol. 137(Pt. D, 2), pp. 77–83, Mar. 1990.

(9) J. R. Sveinsson and F. W. Fairman, "Minimal balanced realization of transfer function

matrices using markov parameters'', *IEEE Trans. Automat. Contr.*, Vol. AC-30(10), pp. 1014–1016, Oct. 1985.
(**10**) C. P. Therapos, ''Balanced minimal realization of SISO systems'', *El. Letters*, Vol. 19(11), pp. 424–426, May 1983.
(**11**) C. P. Therapos, ''Internally balanced minimal realization of discrete SISO systems'', *IEEE Trans. Automat. Contr.*, Vol. AC-30(3), pp. 297–299, Mar. 1985.
(**12**) C. P. Therapos, ''Minimal realization of transfer function matrices via one orthogonal transformation'', *IEEE Trans. Automat. Contr.*, Vol. AC-34(8), pp. 893–895, Aug. 1989.
(**13**) C. P. Therapos, ''Balanced realization of stable transfer function matrices'', *IEEE Trans. Automat. Contr.*, Vol. AC-37(2), pp. 281–285, Feb. 1992.
(**14**) F. B. Yeh and C. D. Yang, ''A simple algorithm on minimal balanced realization for transfer function matrices'', *IEEE Trans. Automat. Contr.*, Vol. 34(8), pp. 879–882, Aug. 1989.
(**15**) A. J. Laub, ''Computation of balancing transformations'', JACC FA8-E, 1980.
(**16**) L. C. Calvez, P. Vilbé and M. Sévellec, ''Efficient evaluation of model reduction related integrals via polynomial arithmetic'', *El. Letters*, Vol. 28(7), pp. 659–661, Mar. 1992.
(**17**) L. C. Calvez, P. Vilbé, A. Derrien and P. Brehonnet, ''General orthogonal sequences via a Routh-type stability array'', *El. Letters*, Vol. 28(19), pp. 1791–1792, Sept. 1992.
(**18**) A. Derrien, C. Nouet, P. Vilbé and L. C. Calvez, ''Orthogonal sets for efficient model-order reduction via a Gauss–Newton method'', *El. Letters*, Vol. 30(7), pp. 544–545, Mar. 1994.
(**19**) S. K. Abdali, B. F. Caviness and A. Pridor, ''Modular polynomial arithmetic in partial fraction decomposition'', in ''Proc. 1977 MACSYMA User's Conf.'', NASA Publication CP-2012, 1977, pp. 253–261.
(**20**) R. A. Roberts and C. T. Mullis, ''Digital Signal Processing'', Addison-Wesley, 1987, pp. 402–403.

Appendix—Modular Polynomial Arithmetic

The efficiency of this block balancing algorithm is partly due to the intensive use of the modular polynomial arithmetic: all the computations are performed with low polynomial degrees.

The orthogonalization matrix C in Eq. (18) is easily obtained by an iterative divide process which has been previously suggested in (**19**). All the computations are performed in an efficient manner since the polynomial degrees of the Laplace or z transforms of the orthogonal functions are reduced using modular polynomial arithmetic.

Consider the block-factorized function given as:

$$F(q) = \frac{N(q)}{D(q)} = \prod_{k=1}^{m} \frac{N_k(q)}{D_k(q)} \quad (\deg [N] \leqslant \deg [D])$$

where $\{D_k; k = 1, \ldots, m\}$ is a set of pairwise relative prime polynomials. The polynomials $P_k(q)$, $k = 1, \ldots, m$ satisfying

$$F(q) = g + \sum_{k=1}^{m} \frac{P_k(q)}{D_k(q)}; \quad g \in \mathbf{R}, \quad \deg [P_k] < \deg [D_k] = n_k$$

can be written, using the notation of modular polynomial arithmetic

$$P_k = \left[\frac{\left(\prod_{k=1}^{m} N_k \right) \bmod D_k}{\left(\prod_{\substack{k'=1 \\ k' \neq k}}^{m} D_{k'} \right) \bmod D_k} \right] \bmod D_k. \tag{A1}$$

Let us introduce $U = (\Pi_{k=1}^{m} N_k) \bmod D_k$ and $V = (\Pi_{k'=1,k'\neq k}^{m} D_{k'}) \bmod D_k$. Then, the P_k are given by:

$$P_k = [(U \bmod D_k)X] \bmod D_k$$

where $X = (1/V) \bmod D_k$ is a polynomial which can be computed via the Euclidean algorithm (**16**).

It is easily seen that if $\deg[N] < \deg[D]$, then $g = 0$, else g can be computed as:

$$g = \prod_{k=1}^{m} g_k \quad \text{with} \quad g_k = \begin{cases} 1 & \text{if } \deg[N_k] < \deg[D_k] \\ \mathrm{quot}(N_k, D_k) & \text{if } \deg[N_k] = \deg[D_k] \end{cases}$$